ON DIRECTED ZERO-DIVISOR GRAPHS OF FINITE RINGS

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Abstract. For an artinian ring R, the directed zero-divisor graph $\Gamma(R)$ is connected if and only if there is no proper one-sided identity element in R. Sinks and sources are characterized and clarified for finite ring R, especially, it is proved that for *any* ring R, if there exists a source b in $\Gamma(R)$ with $b^2 = 0$, then $|R| = 4$ and $R = \{0, a, b, c\}$, where a and c are left identity elements and $ba = 0 = bc$. Such a ring R is also the only ring such that $\Gamma(R)$ has exactly one source. This shows that $\Gamma(R)$ can not be a network for any ring R.

1. Introduction

For any noncommutative ring R, let $Z(R)$ be the set of (one-sided) zero-divisors of R. The directed zero-divisor graph of R is a directed graph $\Gamma(R)$ with vertex set $Z(R)^* = Z(R) - \{0\}$, where for distinct vertices x and y of $Z(R)^*$ there is a directed edge from x to y if and only if $xy = 0$ ([6]). This is a generalization of zero-divisor graph of commutative rings. The concept of a zero-divisor graph of a commutative ring was introduced in [4], and it was mainly concerned with colorings of rings there. In [2], the vertex set of $\Gamma(R)$ is chosen to be $Z(R)^*$ and the authors study interplay between the ring-theoretic properties of commutative ring R and the graph-theory property of $\Gamma(R)$. The zero-divisor graph of a commutative ring is also studied by several other authors, see [3] for a list of references. The zero-divisor graph has been also introduced and studied for semigroups in [5].

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In this paper, we study the directed zero-divisor graph of noncommutative rings and we focus our attention on finite rings (Most results on finite rings in this paper actually holds for artinian rings.). In section 2, we prove that an artinian ring R has connected zero-divisor graph if and only if one-sided identity of R (if exists) is two-sided, if and only if $\Gamma(R)$ contains no end vertex (i.e., sinks and sources). For any distinct vertices x, y of a finite ring with proper one-sided identity, the directed distance from x to y is less than 7, if a directed path exists from x to y. In section 3 and 4, we study sinks and sources of finite and infinite ring. We proved that for *any* ring R, if there exists a source b in $\Gamma(R)$ with $b^2 = 0$, then $|R| = 4$ and $R = \{0, a, b, c\}$, where a and c are left identity elements and $ba = 0 = bc$. Such a ring R is also the only ring such that $\Gamma(R)$ has exactly one source. The dual result for sink vertex is also true. This result is key to clarify sinks and sources in $\Gamma(R)$. In Section 3, we show that for a finite ring R with at least five elements, sink and source can not coexist in $\Gamma(R)$. For finite ring R with at least five elements, sinks (sources) are characterized by strongly right (left) invertible elements. In section 4, we show that for any ring R with at least five elements, there are only four possibilities for the semigroups (when non-empty) $Sink(R)$, the set of sinks of $\Gamma(R)$, and $Sour(R)$, the semigroup of all sources:

(1) $Sink(R) = \varnothing$ and $Sour(R) = \varnothing$;

(2) $Sink(R) = \varnothing$ and $|Sour(R)| \geq 2$;

(3) $Sour(R) = \varnothing$ and $|Sink(R)| \geq 2$;

(4) $|Sour(R)| = \infty = |Sink(R)|$.

Therefore $\Gamma(R)$ can not be a network for any ring R.

2. Connectedness and Diameters

Lemma 2.1. *Suppose that every right identity element of a finite ring R is a two-sided identity. We have*

(1) Each left zero-divisor is also a right zero-divisor;

(2) If in addition $|R| \geq 5$, then for distinct $a, b \in R^*$ with $ab = 0$, there exists $c \in R^*$ such that $ca = 0$ and $c \neq a$.

Proof. (1) If $ab = 0$ and $xa \neq 0$ for all $x \in R$, then $Ra = R$ and there is a right identity element $e \in R$. Then e is also a left identity. Let $ga = e$,then $0 = gab = eb = b$, a contradiction.

(2) Suppose that $ann_l(a) = \{a, 0\}$, then for distinct $b, c, d \in R^* - \{a\}$, we have $daa = 0, da \neq 0$. Thus $da = a$. Then we have $b - c = b - d$ and therefore, $c = d$. Then $|R| \leq 4$. This completes the proof. QED

Lemma 2.2. *Suppose $|R| \leq 4$ and every proper one-sided identity element is a two-sided identity. Then for distinct $a, b \in R^*$ with $ab = 0$, we have $ba = 0$.*

Proof. If R has three elements, then for $a \neq b$ with $ab = 0$, one has $(a + b)b = 0$. Thus $ba = b(a + b) = 0$.

Now suppose that $R = \{0, a, b, a + b\}$ has four elements. Without loss, we assume $ab = 0$. If R has identity element 1, assume $a+b = 1$. Then $a^2 = a, b^2 = b$. So if $ba = 1$, then $a = aba = 0$; if $ba = a$, then $a = a^2 = aba = 0$; If $ba = b$, then $b = b^2 = bab = 0$. There is contradiction in each case. Hence $ba = 0$. In the remaining part of the proof, assume that R has no one-sided identity element. Without loss, we can assume $2a = 2b = 0$ (The only other case is that the additive group of R is cyclic of order four.). Now we show that $ab = 0$ implies $ba = 0$:

(1)$ba \neq a$: If $ba = a$, then $a^2 = 0$. In this case, we assert $b^2 = 0$. Actually, $b^2 \neq b$ since otherwise b is a left identity element of R; $b^2 \neq a$ since otherwise $a = ba = b^2a = a^2 = 0$; $b^2 \neq a + b$ since otherwise, $b^2 = (a + b)b = b^3 = b(a + b) = a + (a + b) = b$. But $b^2 = 0$ implies $a = ba = b^2a = 0$, a contradiction.

(2)$ba \neq b$: If $ba = b$, then $b^2 = 0$. Then $a^2 \neq a$ since otherwise $a(a + b) = a$, $b(a+b) = b$, $(a+b)^2 = a+b$, i.e., $a+b$ is a right identity element of R; $a^2 \neq 0$ since $b = ba^2$;$a^2 \neq b$ since otherwise $b = ba = ba^2 = b^2 = 0$. Finally, $a^2 = a + b$ and we have $a^2 = a + b = a^3 = a^2 + ba = a^2 + b$, contradicting with the assumption $b \neq 0$.

(3) $ba \neq a + b$: If $ba = a + b$, then $a^2 = 0$, $0 = ba^2 = a^2 + ba$, $0 = ba = a + b$, another contradiction. This completes the proof of $ba = 0$, for rings without one-sided identity element. QED

Let K_i be the complete directed graph with i vertices. For any ring R, by [6, Theorem 3.2], there is no isolated vertices in $\Gamma(R)$. Thus by Lemma 2.2, we have a list of all possibilities of $\Gamma(R)$ for rings R with $|R| \leq 4$:

(1) $|R| = 2$: $K_i, i = 0, 1$

(2) $|R| = 3$: $K_i, i = 0, 2$

(3) $|R| = 4$: K_i $(i = 0, 1, 2, 3)$; $\circ \leftrightarrows \circ \leftrightarrows \circ$; $\circ \rightarrow \circ \leftarrow$; $\circ \leftarrow \circ \rightarrow \circ$.

We remark that every graph in the list can be realized as the zero-divisor graph

of some ring R with $|R| \leq 4$. Actually this list is a special case of results in [7, Section 4])

Proposition 2.3. *Suppose that R is a finite ring with the property that every one-sided identity element is a two-sided identity in R. Then for any path $a \to b$ in $\Gamma(R)$, there is a walk $c \to a \to b \to d$, where $c \neq a$ and $b \neq d$.*

A vertex g in a directed graph G is called a *sink*, if the in-degree of g is positive and the out-degree of g is zero. The dual concept of sink is called *source*.

Let $Z_r(R)$ be the set of right zero divisors. In [6], it is proved that for any ring R, $\Gamma(R)$ is connected if and only if $Z_r(R) = Z_l(R)$, i.e., there exists no end-vertex (sink or source) in $\Gamma(R)$. It is also proved that $\Gamma(R)$ is connected for all artinian rings with two sided-identity element. We now characterize all finite rings R whose directed zero-divisor graph $\Gamma(R)$ is connected:

Theorem 2.4. *For any finite ring R, the following statements are equivalent:*

(1) The directed zero-divisor graph $\Gamma(R)$ is connected;

(2) Every one-sided identity element of R is the two-sided identity of R;

(2') Either R has two-sided identity or R has no proper one-sided identity;

(3) There exists no end-vertex (sink or source) in $\Gamma(R)$.([6])

Proof. (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (2). If R has a left identity e that is not a right identity element of R, then there exists $a \in R$ such that $ae - a \neq 0$. In this case $ae - a \to b$ for all $b \in R^*$. Then there exist at least two left identity elements in R, say e and $f = e + ae - a$. Then e and f are sink vertices in $\Gamma(R)$. So the directed graph $\Gamma(R)$ is not connected.

(2) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (1). Assume that every one-sided identity element of R is the two-sided identity of R. If $|R| < 5$, then by Lemma 2.2, we know that there is no sink (source) vertex in $\Gamma(R)$ and that $\Gamma(R)$ is connected. In what follows, we assume $|R| \geq 5$. Then by Lemma2.1, if $a \to b$ for distinct vertices a, b, then there exist $c, d \in R^*$ such that $c \neq a, d \neq b$ and in $\Gamma(R)$ there is a walk $c \to a \to b \to d$. So there is no sink or source vertex in $\Gamma(R)$. This proves (3). Now we use the proof of corresponding result in [2] to finish our proof: For any distinct $x, y \in Z(R)^*$, if $xy = 0$, then $d(x, y) = 1$. If $xy \neq 0$, then again by Lemma 2.1 and [6, Theorem 3.2], there exists $a \neq x, b \neq y$ such that $xa = 0 = by$. If $a = b$,, then $x \to a \to y$ and $d(x, y) = 2$; If $a \neq b$ and $ab = 0$, then we have

$x \to a \to b \to y$ and $d(x, y) \leq 3$; If $a \neq b$ and $ab \neq 0$, then $ab \neq x, ab \neq y$ since $xy \neq 0$, and there is a path $x \to ab \to y$. In all cases, $d(x, y) \leq 3$. QED

We remark that Theorem 2.4 actually holds for a still wider class of rings - rings which are both right artinian and left artinian, since Lemma 2.1 holds for artinian rings.

Now let R be a ring with proper left identity element e. Denote

$$I_e = \{a \in R | ae = 0\}, R_e = \{a \in R | a = ae\}.$$

Then (1) $I_e = ann_l(e)$ and I_e is a two-sided ideal of R with at leat two elements;

(2) R_e is a subring of R with identity e;

(3) $R = R_e \oplus I_e$ as left R-module;

(4) There is a ring isomorphism $R_e \cong R/I_e$.

For any subset M, N of $Z(R)^*$, let $\Gamma(M, N)$ be the induced bipartite subgraph of $\Gamma(R)$ and denote its edge set as $E(M, N)$. For the graph $\Gamma(R)$, denote its edge set as $E(R)$. If there is a cycle, we add 2 to the number of the edges of $\Gamma(R)$. Then the graph $\Gamma(R)$ is completely determined by the following four induced subgraphs: $\Gamma(R_e)$, complete directed graph $\Gamma(I_e)$, bipartite graphs $\Gamma(I_e^*, R_e^*)$ and $\Gamma(R_e^*, R_e * \oplus I_e^*)$. We have

Proposition 2.5. *For a ring R with proper left identity element e, the graph $\Gamma(R)$ has the following properties:*

(1) For any $a \in I_e$, the out-degree of a is $|R| + 1$;

(2) The number of vertices of $\Gamma(R)$ is $|R| - 1 = |R_e||I_e| - 1$;

(3) $|E(R)| = |I|[|I| - 1 + E(R_e^*, I_e^*) + E(R_e^*, R_e^* \oplus I_e^*) + (2 - |I|)E(R_e)]$.

Proof. (1) The number of directed edges from I_e^* to R^* is $|R|(|I| - 1)$;

(2) The number of directed edges from $R^* - I_e^*$ to I_e^* is

$$|I|[|E(R_e^*, I_e^*)| - (|I| - 1)(|R_e| - 1)];$$

(3) An element of $R^* - I_e^*$ has the form of $a_i + x$, where $x = 0$ or $x = b_j$. Besides, $(a_i + x)(a_j + y) = a_i(a_j + y)$. Thus the number of directed edges from $R^* - I_e^*$ to $R^* - I_e^*$ is

$$|I|[|E(R_e)| + |E(R_e^*, R_e^* \oplus I_e^*)| - (|I| - 1)|E(R_e)|].$$

Note that $|R| = |R_e||I_e|$, then we obtain our formula. QED

Proposition 2.6. *Let R be a finite ring with proper one-sided identity. For distinct elements x, y in R, either $d(x, y) = \infty$ or $d(x, y) \leq 6$.*

Proof. Let R be a ring with proper left identity and assume that e is a left identity of R. Then $R^* = K \cup I_e^* \cup R_e^*$ and this is a disjoint union where we assume

$$R_e^* = \{a_i | i = 1, 2, \cdots, m\}, I_e^* = \{b_j | j = 1, 2, \cdots, n\}$$

and $K = I_e^* \oplus R_e^*$. For distinct $x, y \in R^*$, assume that there exists a directed path from x to y in $\Gamma(R)$, say, $x \to x_1 \to x_2 \to \cdots \to x_r \to y$.

(1) If $x \in I_e$, then $x \to z$ for any z. In this case, $d(x, y) = 1$.

(2) Assume $x \in R_e$. If $0 \in xI_e^*$, then we have a path $x \to I_e^* \to y$, $d(x, y) \leq 2$; If $0 \notin xI_e^*$, then $0 \notin xK$. Without loss, we can assume $x_i \in R_e$ for $i = 1, 2, 3, 4$. Since R_e is a ring with two-sided identity, $diam(R) \leq 3$. So there exists a path $x \to x_1 \to x_2 \to x_3 \to I_e^* \to y$. In this case, $d(x, y) \leq 5$.

(3) The final case is $x \in K$. Assume $x = a_1 + b_1$. If $0 \in xI_e^*$, then $d(x, y) \leq 2$; If $0 \notin xI_e^*$, then $x_1 \in R_e^* \cup K$. If $x_1 \in K$, then we have $0 = (a_1 + b_1)(a_s + b_t) = a_1 a_s + a_1 b_t$. Then we have a path $x \to b_t \to y$, and $d(x, y) \leq 2$; Hence we can assume $x_i \in R_e^*$ for $i = 1, 2, 3, 4$. and $x_4 \to I_e^*$. We have a path $x \to x_1 \to x_2 \to x_3 \to x_4 \to I_e^* \to y$ and hence $d(x, y) \leq 6$. This completes the proof. QED

Corollary 2.7. *For any ring R with proper one-sided identity element e, R_e is a subring of R and e is a two-sided identity of R_e, and $diam(R) \leq 3 + diam(R_e)$.*

We end this section with the following examples:

Example 2.8 For any field F, let R be the n by n full matrix ring over F ($n > 1$). Then the diameter of $\Gamma(R)$ is 2.

We need only to prove the following facts: for any $A, B \in Z(R)^*$, there exists $C \in Z(R)^*$ such that $AC = 0$ and $CB = 0$. First, there exist invertible matrices $P, Q \in R$ such that the last column of AP is zero and the first row of QB is zero. Second, let

$$C = P \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} Q,$$

then we have $C \neq 0$, $AC = 0, CB = 0$.

Example 2.9 Let R be the n by n full matrix ring over $\mathbb{Z}/2\mathbb{Z}$. Let S be the non-unitary subring of R consisting of those matrices all of whose rows are zero except the first row. Then $Sink(S) = \{\begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} \mid \alpha\}$, and $Sink(S)$ consists of all left identity elements of S. $I_e = \{\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \mid \alpha\}$ for $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. $|S| = 2^n, |I_e| = |Sink(S)| = 2^{n-1}$. $\Gamma(S)$ is a star-like directed graph whose kernel is the complete graph $K_{2^{n-1}-1}$, each vertex of which also connects to 2^{n-1} sinks.

Example 2.10 Let $R = \mathbb{Z}/n\mathbb{Z}$ and denote $S = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in R\}$. Then $Sink(S) = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in R, a = \overline{m}, (m, n) = 1\}$. $I_e = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R\}$ for $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus $|Sink(S)| = n\varphi(n)$, $|I_e| = n$, where $\varphi(n)$ is the Eulerian number of n. When $n \geq 3$, there is no source in $\Gamma(R)$. The clique number of $\Gamma(R)$ is $n - 1$.

3. Sinks and Sources of finite rings

Proposition 3.1. *(1) For any ring R, if there exists a source b in $\Gamma(R)$ with $b^2 = 0$, then $R = \{0, a, b, c\}$, where a and c are left identity elements, $ba = 0 = bc$;*

(2) For any ring R, if there exists a sink b in $\Gamma(R)$ with $b^2 = 0$, then $R = \{0, a, b, c\}$, where a and c are right identity elements, $ab = 0 = cb$.

Proof. (1) Let b be a source in $\Gamma(R)$ with $b^2 = 0$. Then consider the left R-module epimorphism $\eta : R \to Rb, r \mapsto rb$. Notice that $ker(\eta) = ann_l(b) = \{0, b\}$ is a submodule of R. For any $r \in R - \{0, b\}$, we have $rb = b$. We conclude $|R| \leq 4$, since otherwise, let $a, c, d \in R - \{0, b\}$ be distinct elements. Then we have $(a - c)b = b - b = 0 = (a - d)b$. Therefore, $a - c = a - d$ and thus $c = d$, a contradiction.

Now by the proof of Lemma 2.2 and Theorem 2.4, R has proper left identity element. So R contains at least two left identity element, say a and c. Thus $R = \{0, a, b, c\}$, where a and c are left identity elements, $ba = 0 = bc$. In this case, we have $a \leftarrow b \leftrightarrows b \to c$.

(2) The proof is dual to the above proof.

The structure of rings R with $|R| \leq 4$, as well as the related graph $\Gamma(R)$, is

rather clear (see Lemma 2.2 and the listing following Lemma 2.2). As for finite rings R with $|R| \geq 5$, we have the following:

Proposition 3.2. *Let R be a finite ring with proper left identity elements. If $|R| \geq 5$, then*

(1) $\Gamma(R)$ contains at least two sinks;

(2) For any sink r in $\Gamma(R)$, $r^2 \neq 0$;

(3) $\Gamma(R)$ contains no sources.

Proof. Since every left identity element of R is a sink in $\Gamma(R)$, thus $\Gamma(R)$ has at least two sinks. Let e be any left identity element of R. Then I_e^* is not empty, and for any $x \in I_e^*, y \in R^*$, we have $xy = 0$. Thus every non-zero element of R is a vertex of $\Gamma(R)$ and, if $\Gamma(R)$ has source, the source vertex must lies in I_e. Of course, $\Gamma(R)$ has source vertex if and only if $|I_e| = 2$ and there is no directed edge from $R^* - I_e^*$ to I_e^*. This is equivalent to saying that for the left identity element e, there exists a non-zero element b of R, such that $ann_l(e) = \{0, b\}$ and $ann_l(b) = \{0, b\}$. So if $\Gamma(R)$ has a source, this source is b. Then we have $b^2 = 0$, contradicting with the assumption and Proposition 3.1. With the same reason, sinks $r \in \Gamma(R)$ must satisfy $r^2 \neq 0$. QED

Corollary 3.3. *Let R be a finite ring with proper left identity elements. Assume that R has at least five elements. For any left identity e of R, if $ann_l(e) = \{0, b\}$ for some non-zero element b, then in $\Gamma(R)$, the out-degree of b is $|R| - 1$ and the in-degree of b is positive.*

Similarly, we have

Proposition 3.4. *Let R be a finite ring with proper right identity elements. If R has at least five elements, then*

(1) $\Gamma(R)$ contains at least two sources;

(2) For any source r in $\Gamma(R)$, $r^2 \neq 0$;

(3) $\Gamma(R)$ contains no sink.

Corollary 3.5. *Let R be a finite ring with proper right identity elements. If R has at least five elements, then for any right identity e of R with $ann_l(e) = \{0, b\}$ for some non-zero element b, the in-degree of b in $\Gamma(R)$ is $|R| - 1$ and the out-degree of b is positive.*

Recall that a *network* N is a directed graph with exactly one sink vertex k and a unique source vertex c such that c connects to every vertex of N and every vertex of N connects to k. By Lemma 2.2, Theorem 2.5, Propositions 3.1,3.5 and 3.7, we immediately have,

Corollary 3.6. *For any finite ring R with at least five elements,*

(1) $\Gamma(R)$ can not contain sink and source at the same time;

(2)([7, Cor. 3.2]) $\Gamma(R)$ is not a network for any finite ring.

Proof. (2) is an obvious consequence of (1). To prove (1), we list all our previous results on rings with proper one-sided identity in a single place as follows:

(a) If $|R| \le 4$, then $\Gamma(R)$ is one of the following

$$\circ \leftarrow \circ \rightarrow \circ, \circ \rightarrow \circ \leftarrow \circ.$$

(b) If $|R| \ge 5$ and R has proper left identity elements, then $\Gamma(R)$ contains at least two sinks, but in it there is no source.

(c) If $|R| \ge 5$ and R has proper right identity elements, then $\Gamma(R)$ contains at least two sources, but in it there is no sink.

Definition 3.7 Suppose that R has proper left (respectively, right) identity element. An element $r \in R$ is called **strongly right (left) invertible**, if for *any* left (right) identity element e of R, r has unique right (left) inverse s relative to this e. In this case, such s has more than one left (right) inverse relative to the same left identity e. Obviously, r is strongly right invertible if for *some* left identity element e of R, r has a unique right inverse s relative to this e.

Proposition 3.8. *For an element r in a finite ring R satisfying $r^2 \neq 0$, r is a sink vertex (source vertex) in $\Gamma(R)$, if and only if r is strongly right (respectively, left) invertible in R.*

Proof. (1) Suppose r is a sink vertex of $\Gamma(R)$. We have $ry \neq 0$ for all $y \in R^*$ since $r^2 \neq 0$. Since R is a finite ring, we have $rR = R$. Assume $re = r$. Then from $res = rs \neq 0$ for all $s \in R^*$, we obtain $es = s$ for all $s \in R$. Thus e is a left identity of R. For any left identity $f \in R$, let $ru = f$. Then u is unique relative to the f, since r is a sink vertex. For the same reason, there is $r \neq v \in R^*$ such that $vr = 0$. Hence $0 = vru = vf$. Thus f is not a right identity element of R. In

this case, we have $(r+v)u = ru + v(fu) = f$, where $r+v \neq r$. Thus this u has at least two left inverses (say, r and $r+v$) relative to the left identity f. Thus r is strongly right invertible in R. Notice that R has at least two sink vertex in this case.

(2) Conversely, Suppose that r is strongly right invertible in R. Then by definition, there is left, but not right, identity element in R, and for any left identity $e \in R$, r has a unique right inverse u relative to this e. If there is a path $r \to v$, then we have $r(u+v) = e$ and this implies $v + u = u$, a contradiction. Finally, suppose $x = ae - a \neq 0$, then $x \neq r$ and $xr = 0$. Thus r is a sink vertex of $\Gamma(R)$.

The proof of the other case is similar. QED

Corollary 3.9. *Let R be a finite ring.*

(1)If $\Gamma(R)$ contains exactly one source (respectively, sink), then $|R| = 4$, and R^* has the form of

$$a \leftarrow b \to b \to c;$$

$$(respectively, a \to b \to b \leftarrow c);$$

(2)Let $|R| \geq 5$. Then an element r of R^* is a sink (source) if and only if r is strongly right (left) invertible in R. In this case, $r^2 \neq 0$ and in $\Gamma(R)$ there are at least two sinks (sources).

4. Sink(R), Sour(R) and network

All rings in this section have at least five elements.

Definition 4.1 For any ring R, denote

$Sink(R) = \{$sinks in $\Gamma(R)$ $\}$, $Sour(R) = \{$sources in $\Gamma(R)$ $\}$,

$Inv_r(R) = \{$ strongly right invertible elements of R relative to some proper left identity $\}$,

$Inv_l(R) = \{$ strongly left invertible elements of R relative to some proper right identity$\}$.

Proposition 4.2. *Let R be any ring with at least five elements.*

(1) If $Sink(R)$ (respectively, $Sour(R)$) is not empty, then it is a left (right) cancellative multiplicative semigroup;

(2) If $Inv_r(R)$ ($Inv_l(R)$) is not empty, then it is also a left (right) cancellative multiplicative semigroup and $Inv_r(R) \subseteq Sink(R)$ ($Inv_l(R) \subseteq Sour(R)$);

(3) $Sink(R) = Z_r(R) - Z_l(R)$*,* $Sour(R) = Z_l(R) - Z_r(R)$*;*

(4) $Z(R)^*$ *has a disjoint decomposition*

$$Z(R)^* = Sour(R) \cup (Z_r(R) \cap Z_l(R)) \cup Sink(R).$$

Proof. (1) Assume that $Sink(R)$ is nonempty. For any $a, b \in Sink(R)$, there exists $x \in R^*$ such that $x \neq a$ and $xa = 0$. By Proposition 3.1, $ann_r(a) = 0$. So $Sink(R)$ is left cancellative and $Sink(R) = Z_r(R) - Z_l(R)$. We assert $x \neq ab$, since otherwise, we would have $abx = 0$, which implies $x = 0$. Thus $ab \in Sink(R)$, since $xab = 0$. Hence $Sink(R)$ is a semigroup under the multiplication of R.

(2) Suppose that $Inv_r(R)$ is not empty. For any proper left identity elements e, f of R and any $a, b \in Inv_r(R)$, let $ax = e, f = by$. Then $(ab)(yx) = afx = ax = e$. Thus ab is right invertible. If $(ab)c = e = (ab)d$, then $bc = bd$, and $c = d$ since $Inv_r(R) \subseteq Sink(R)$. Thus ab is strongly right invertible, i.e., $ab \in Inv_r(R)$ and hence $Inv_r(R)$ is a semigroup.

The other case is dual to the sink case. QED

Remark For some fixed proper left identity e, let $Inv_{r_e}^{-1}(R) = \{u \in R | au = e$ for some $a \in Inv_r(R)$ $\}$. Then it is easy to verify that $Inv_{r_e}^{-1}(R)$ is a multiplicative semigroup with identity e.

Proposition 4.3. *For any ring* R *with at least five elements,* R *contains a proper left identity if and only if the following two conditions hold: (1)* $Sink(R) \neq \varnothing$*; and (2) There exists an* $x \in Sink(R)$ *such that* $x[Sink(R)] = Sink(R)$*.*

In this case, $\Gamma(R)$ *contains no source while* $Sink(R)$ *contains at least two elements.*

Proof. If R contains a proper left identity e, then $e \in Sink(R)$ and $e[Sink(R)] = Sink(R)$.

Conversely, assume that $Sink(R) \neq \varnothing$ and $x[Sink(R)] = Sink(R)$ for some $x \in Sink(R)$. Let $xe = x$, $e \in Sink(R)$. Then $x(ey - y) = 0$ for all $y \in R^*$. Then by Proposition 3.1, $ey = y$ and hence, e is a left identity of R. Since e is a zero-divisor, it is a proper left identity.

If there is a source in $\Gamma(R)$, then it must lie in I_e. Then $I_e = \{0, b\}$ for some nonzero b. Then b is a source of $\Gamma(R)$ with $b^2 = 0$. Then by Proposition 3.1, $|R| = 4$ and there is no sink in $\Gamma(R)$, a contradiction. So in this case, there is no source in $\Gamma(R)$. QED

Since $Sink(R)$ is left cancellative, we immediately have

Corollary 4.4. *For any ring R with at least five elements, if $0 < |Sink(R)| < \infty$, then $Sink(R) = Inv_r(R)$ and there is no source in $\Gamma(R)$.*

The following two results are duals of 4.3 and 4.4

Proposition 4.5. *For any ring R with at least five elements, R contains a proper right identity, if and only if the following two conditions hold: (1) $Sour(R) \neq \varnothing$; and (2) There exists an $y \in Sour(R)$ such that $[Sour(R)]y = Sour(R)$.*

In this case, $\Gamma(R)$ contains no sink but it contains at least two sources.

Corollary 4.6. *For any ring R with at least five elements, if $0 < |Sour(R)| < \infty$, then $Sour(R) = Inv_l(R)$ and there is no sink in $\Gamma(R)$.*

As a combination of Propositions 4.2, 4.3 and 4.5, we have

Proposition 4.7. *For any ring R, one-sided identity of R are two-sided identity if and only if $\Gamma(R)$ satisfies one of the following conditions:*

(1) $Sink(R) = \varnothing$ and $Sour(R) = \varnothing$;

(2) $Sink(R) = \varnothing$, $Sour(R) \neq \varnothing$ and for any $t \in Sour(R)$, $[Sour(R)]t \subset Sour(R)$; (In this case, $\Gamma(R)$ has infinitely many sources);

(3) $Sour(R) = \varnothing$, $Sink(R) \neq \varnothing$ and for any $s \in Sink(R)$, $s[Sink(R)] \subset Sink(R)$; (In this case, $\Gamma(R)$ has infinitely many sinks);

(4) $Sink(R) \neq \varnothing$ and for any $s \in Sink(R)$, $s[Sink(R)] \subset Sink(R)$. At the same time, $Sour(R) \neq \varnothing$ and for any $t \in Sour(R)$, $[Sour(R)]t \subset Sour(R)$. (In this case, $\Gamma(R)$ has infinitely many sinks and infinitely many sources.)

Corollary 4.8. *Suppose that in a ring R, one-sided identity element is two-sided identity. If in addition, R satisfies descending chain condition on principal left (respectively, right) ideals, then $\Gamma(R)$ contains no source (sink). In particular, if R is a left and right artinian ring, and one-sided identity element in R is two-sided identity, then $\Gamma(R)$ contains neither source no sink.*

Proof. It is easy to verify that for $K = Sink(R)$ and any $t \in K$, $tK \subset K$ if and only if for any (or some) $s \in K$, $stK \subset sK$, if and only if $stR \subset sR$. So, if R has DCC on right principal ideals, then in Proposition 4.7, cases (3) and (4) could not occur. Thus $\Gamma(R)$ contains no sinks. QED

Finally, as a corollary of Propositions 3.1, 4.3, 4.5 and 4.7, we have

Corollary 4.9. *For any ring R, $\Gamma(R)$ is not a network.*

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